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13. ABSTRACT (Maximum 200 words)

There were three major research explorations.

1. Wavelets: Necessary and sufficient conditions on the wavelet, scaling function and projection kernel for given rates of convergence of wavelet expansions in the supremum and $L^p(R^d)$ norms have been given.

2. Image Compression and Reconstruction: A new algorithm for image compression is developed using quasi-interpolant multivariate box splines and multi-resolution analysis has been developed.

3. Shallow Water Theory: A mathematical justification for the "shallow water theory for time-dependent two-dimensional flows of an inviscid, irrotational, incompressible fluid moving.under the influence of gravity has been developed.

14. SUBJECT TERMS
1. Wavelet expansions; rates of convergence, supremum norm.
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3. Shallow water, inviscid, irrotational incompressible fluid.

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FINAL TECHNICAL REPORT TO AFOSR

JUNE 15, 1991- JANUARY 14, 1995

AFOSR CONTRACT #NM 91NM062 with SUBCONTRACT to BOSTON UNIVERSITY

WAVELETS AND SPLINES IN NUMERICAL METHODS AND IMAGE COMPRESSION

DEPARTMENT OF MATHEMATICS
HOWARD UNIVERSITY

I. RESEARCH OBJECTIVES AND STATUS OF RESEARCH
- A. Wavelet research
  - A 1. Pointwise Convergence of Wavelet Expansions
  - A 2. Rates of Convergence of Wavelet Expansions
  - A 3. Quasiorthonormality and Wavelets
  - A 4. Classification of Noises via Autocorrelation
- B. Compression of Images via a Quasi-Interpolation Multiresolution Box-Spline Framework
- C. Shallow Water Theory
- D. Weiner-Wintner Theorem in Higher Dimensions
- E. Interactions with ComSERC and Howard University Researchers

II. PUBLICATIONS 1991-1995

III. PROFESSIONAL PERSONNEL AND CONSULTANTS
- A. Full Time
- B. Consultants
- C. Graduate Students

IV. PAPERS PRESENTED AT CONFERENCES AND SEMINARS

V. SEMINARS/COLLOQUIUMS AT HOWARD UNIVERSITY

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REPORT SUBMITTED BY: LOUISE A. RAPHAEL, P.I. Louise A Raphael

DANIEL A. WILLIAMS, CO-P.I. Daniel A. Williams

MARCH 15, 1995

INTRODUCTION.

In this final report of our three and half years research sponsored by the Air Force Office of Scientific Research, we provide a comprehensive list of our research activities, accomplishments and work in progress during the period of June 15,1991 to January 14,1995.

There were two major research explorations by the Howard University wavelets group. The first was in the pointwise and $L^p(R^d)$ convergence and rates of convergence for multiresolution approximations and wavelet expansions. Our second major emphasis was image and curve processing, compression and restoration by a quasi-interpolation multivariate spline method. In addition several other research projects including wavelets results are also noted in this report.

Our papers including theoretical as well as algorithmic papers have been published and accepted in 17 leading research journals and proceedings, with four other papers submitted and two in preparation. In addition the PI Raphael was an editor on the critically acclaimed book Wavelets and Its Applications [Be et al].

Presently we are continuing our research in image processing, classification and neural networks.

I.A. Wavelets Research

Mark A. Kon, Subcontractor, Boston University
Louise A. Raphael, PI, Howard University

Wavelets have been successfully applied to signal processing, numerical analysis, and quantum mechanics. They are especially useful as they have both good time and frequency localizations, as well as regularity (differentiability) and approximation properties

I.A 1. **Pointwise Convergence of Wavelet Expansions**

Much interest has been shown in our joint research with S.E. Kelly on pointwise and $L^p(R^d)$ convergence of wavelet expansions results [KKR1, KKR2, KKR3]. We have shown under general hypotheses on the scaling functions and/or wavelet, that for general functions in $L^p(R^d)$ for $d \geq 1$, that multiresolution approximations $(1<p<\infty)$, scaling - wavelet expansions $(1<p<\infty)$, and wavelet expansions $(1<p<\infty)$ of the form $f(x) = \Sigma c_{jk} w(2^j x-k)$, $j \in Z$, $k \in Z^d$ are shown to converge pointwise almost everywhere and in L^p norm. As an immediate corollary we have that the best L^2 spline approximations on uniform meshes converge pointwise almost everywhere. The proof for non-uniform meshes is still an open problem.

The essential ideas of the proof are that, unlike the Fourier summation kernel associated with the projection operators of L^2 onto the scaling spaces, the wavelet summation kernels are bounded by radial decreasing $L^1(R^d)$ convolution kernels. In the case where the existence of only the wavelet is assumed (i.e., the existence of the scaling function is not assumed) cancellation of the projection kernel (from L^2 onto L^2) off the main diagonal is proved. Namely the kernel of the positive scale portion is shown to be the same as negative of the kernel of the positive scale part. We also show that the summation of wavelet expansion is partially insensitive to order of summation.

I.A 2. Rates of Convergence of Wavelet Expansions

In [KR1, KR2] necessary and sufficient conditions on the wavelet, scaling function, projection kernel and symbol for given rates of convergence of wavelet expansions in the supremum norm are given. For example, such expansions have order of convergence s if and only if the basic wavelet is in the homogeneous Sobolev space of order (-s-d/2), where d is the dimension of the space. These results hold in one and in multiple dimensions.

In [KR3] we used similar techniques to prove that that spline expansions converge almost everywhere to any L^2 function being approximated on grids which are uniform or with bounded mesh ratios. We also give rates of convergence of such expansions in L^{∞} as the mesh size goes to 0.

I.A 3. Quasiorthonormality and Wavelets

Jean Paul Pemba, Postdoc, Howard University

Recent work introduces the concept of quasiorthonormal basis in a Hilbert space. Its properties are studied and applications to wavelets and multiresolution analysis in $L^2(\mathbb{R})$ are proved in [P2] and [P4].

I.A 4. Classification of Noise via Autocorrelation

John Benedetto, Consultant, University of Maryland
Rodney Kerby, Researcher, Howard University
Louise Raphael, PI, Howard University
Daniel Williams, co-PI, Howard University

The primary goal of this ongoing project is to classify and analyze various types of perceived noises. The mathematical definition of certain noises, such as white noise or Weiner processes, obviate the subtlety of properly defining and classifying noises.

Our basic techniques involve extension and modifications of successful noise suppression methods from generalized harmonic analysis and from wavelet and frame theory . These include methods of Weiner and Rice, Benedetto and Teolis, Daubechies, and Wornel and Oppenheim. We are extending these standard techniques in a new way to deal with a variety of noises such as cockpit disturbances, traffic, reverberation, shot effect.

An experimental component of our approach is to model mathematically the correlation of candidates for noise. Such candidates are characterized by properties which hinder clear reception of given classes of signals without further processing. We use arithmetic mean correlations because of the difficulty in directly analyzing the noise. An analytic component of our work is the construction of a bounded function F with noise-like properties whose correlation vanishes at "infinity" so that pure tones are not present. Existing constructions and qualitative examples using probabilistic and ergodic methods are inadequate for these purposes.

Our functions also have the property that the wavelet system $\{F_{m,n}\}$ is uncorrelated over dilations. This ensures a way of building various noises, and provides a method of locally detecting signals in noise. The mathematical setting is obviously not in the context of finite energy signals. Successful completion of this analysis will lead to an experimental tool, viz., two dimensional color coded wavelet plots measuring local noise intensity and phase for noises. A mathematical problem which rounds out the project is to quantify noises in terms of the power spectra which can be computed by the aforementioned methods. Preliminary results have been obtained.

B. Compression of Images via a Quasi-Interpolation Multivariate Box-Spline Framework

Harvey Diamond, Consultant, West Virginia University
Louise A. Raphael, PI, Howard University
Daniel A. Williams, III, co-PI, Howard University

Mallat et al. have designed compact image coding algorithms that separate the edge from the texture information of a photograph. The essential key is to detect multiscale edges from the local maxima of the wavelet transform modulus. They then reconstruct an image approximation from the important edges that are selected by the coding algorithm. An error image is computed by subtracting the reconstructed image from the original one. This error image contains the texture information, which is coded with an orthonormal basis. They have achieved high compression rates.

Aldroubi and Unser (at NIH) have used spline wavelets for their work, while Lucier and deVore have impressive data compression and reconstruction rates with respect to L^1 norm. These results seem to indicate that wavelets are accurate, however reconstruction in real time is a problem. Moreover there is a need for baseline work in splines to which these wavelet reconstructions can be compared. Our prototypical work shows that our multivariate box spline algorithm is more accurate than originally intuited by others.

The algorithm encodes the data of a picture file as a linear combination of integer translates of a box spline. A key feature is the two scale relation $B(x) = \sum c_{jk} B(2x - k)$ which is used extensively for decomposition purposes. Ours is truly a two-dimensional scheme, while many methods scan line by line. The latter schemes consider a picture as an array of pixels on a single row. They record shades of gray along a scan line and look for

discontinuities or jumps in order to detect edges. As we use box splines our edge detection scheme does not scan .pixel by pixel. The key is we look at differences of intensities in a two dimensional sense.

Our alternative approach of using multivariate box-splines and approximate" orthogonal projections was introduced [DRW1] to give a numerical method for Poisson's equation on a square. Our tables showed a significant gain in accuracy and stability.

In [DRW2, DRW3] our technique is to represent the intensity function of the image by a finite sequence of quasi-interpolant approximations using multivariate box-splines at successively halved resolutions. This is a local method and contrasts with spline and wavelet methods which employ exact orthogonal projections.

This multivariate box-spline quasi-interpolation approximation is especially effective in image regions where the pixel values vary smoothly. In regions where there are edges, our algorithm retains more coefficients in order to delineate the edges. This results in a good approximation of the original picture.

In [DRW3] we applied our above algorithm to smooth out noisy curves. This has applications to forged handwriting and removing noise in medical signals.

C. Shallow Water Theory

James E. Donaldson, Howard University

Daniel A. Williams, co-PI, Howard University

A mathematical justification for the "shallow water" theory for time-dependent two-dimensional flows of an inviscid, irrotational, incompressible fluid moving under the influence of gravity in the space of square integrable measurable functions is provided [DW1, DW2, AW]. In the first paper a new derivation of the shallow water theory is provided which enabled the researchers to extend some of their results to three-dimensional flows.

The results of the first paper naturally lead to the investigation of an abstract two point boundary value problem. This led to the development of techniques published in a second paper [DW2], which were instrumental in the proof of the 3-dimensional problem in the Ph. D . thesis of A. Al-hoori.

D. Weiner-Wintner Theorem in Higher Dimension

Rodney Kerby, Researcher, Howard University

Wiener, in devising a Fourier analysis theory for the space of bounded functions, interrelated the ideas of the average autocorrelation and Fourier transforms. A generalization of this result [K] for a positive Radon measure π on Fourier space has been proved. A function F has been constructed such that the average autocorrelation of F is equal to the Fourier transform of π. There three different constructions are given. For example, one has good control at the jumps, while another construction has good control at specific positions. This result has applications to periodic or random noise signals that exist over the entire space where the energy content is infinite. John Benedetto has been the advisor on this project.

E. Interactions with Howard University Researchers

1. Dr. James Donaldson is a researcher in differential equations and applied mathematics. Donaldson and co-PI Williams have completed research on shallow water theory.

2. Drs. Tepper Gill and Zachary Woodward are principal investigators of major research grants supporting the Howard University Computation Science and Engineering Research Center (ComSERC). Under an Army Research Office grant, ComSERC is part of the University of Minnesota's Army High Performance Computing Research Center. This ARO grant provided Raphael and Williams each with a SUN work station. Williams, the co-PI is serving as a consultant on medical imaging for ComSERC and has transferred some of this expertise to the AFOSR project.

II.PUBLICATIONS 1991 -1995

II.A. PUBLISHED AND ACCEPTED

[Be et al] Gregory Beylkin, Ronald Coifman, Ingrid Daubechies, Yves Meyer and Louise Raphael Wavelets and Their Application, Jones and Bartlett, 1992.

[DiRW1] Harvey Diamond, Louise Raphael and Daniel Williams. Box-Spline-Based Approach to the formulation of Numerical Methods for Partial Differential Equations. Numerical Methods for Partial Differential Equations 8,291-301,1992.

[DiRW2] Harvey Diamond, Louise Raphael and Daniel Williams, *A Quasi-Interpolant Box-Spline Formulation for Image Compression Reconstruction,* Second International Conference on Curves and Surfaces, Chamonix, France,Laurent, Mehaute, and Schumaker, editors, A.K. Peters Publ. 4, pp 221-228,1994,.

[DiRW3] Harvey Diamond. Louise Raphael and Daniel Williams. *Image Compression and Reconstruction Based on a Multivariate Box- Spline Method,* International Proceedings of Advances in Computational Mathematics , New Delhi, India,C. Micchelli and H. P. Dikshit, editors, World Scientific Publishers, pp. 157-170,1994.

[DiRW4] Harvey Diamond, Louise Raphael and Daniel Williams. *Restoring shapes of noisy curves,* Wavelet Applications in Signal and Image Processing II, A. Laine and M. Unser, editors, International Society for Optical Engineering Publishers, pp.356-365,1994.

[DiR] Harvey Diamond and Louise A. Raphael. *Comparison of two elementary* *approximation methods.* Mathematics Magazine, vol 67 (5), pp. 359-365, December 1994.

[DW1] James Donaldson and Daniel Williams. *The Linear Shallow Water Theory: A Mathematical Justification.* SIAM Journal on Mathematical Analysis, 6 (4), 892-910,1993,.

[DW2] James Donladson and Daniel Williams, *An Abstract Two Point Boundary Value Problem,* Proceedings of International Conference on Signal Processing and Mathematical Analysis, Cairo, Egypt, I. Mourad, Z. Nashed, and A. Zayed editors, AMS publishers, accepted for 1995 publication.

[HR] Phil Hirschhorn and Louise Raphael. *The Coalgebraic Foundation of the Method of Divided Differences.* Advances in Mathematics. Vol. 91, no. 1, pp. 75-135, 1992.

[KKR1]. Susan E. Kelly, Mark A. Kon and Louise Raphael, *Pointwise Convergence for Wavelet Expansions,* Bulletin of the American Mathematical Society, 30 (1), 87-94, 1994.

[KKR2] Susan E. Kelly, Mark A. Kon, and Louise A. Raphael. *Convergence: Fourier Series Versus Wavelet Expansions,* Proceedings of NATO 1992 Advanced Study Institute on Wavelets and Their Apoplications, James S. Byrnes, editor, Kluwer Academic Publishers, pp.39-50, 1994.

[KKR3] Susan E. Kelly, Mark A. Kon and Louise Raphael, *Local Convergence for Wavelet Expansions.* Journal of Functional Analysis, vol 126 (1), pp 102-138, Nov. 1994.

[K1] Mark Kon, A general result on the probability 0 limit of probabilistic complexity, Journal of Complexity.

[K2] Mark Kon, Review of Introduction to Algebraic and Constructive Field Theory, by J. Baez, I.E. Segal, and Z. Zhou, Physics Today.

[KR1] Mark Kon and Louise Raphael, *Characterizing Convergence Rates for Multiresolution Approximations,* Proceedings of Neaman Workshop on Signal Processing, Technion, Haifa, Israel, J. Zeevi, ed. accepted for 1995 publication.

[KR2] Mark Kon and Louise Raphael, *Generalized Multiresolution Analysis and Convergence of Spline Approximations on* R^d, Proceedings the International Conference on Signal Processing and Mathematical Analysis, Cairo, Egypt, 1994, I. Mourad, Z. Nashed, and A. Zayed editors, AMS publishers, accepted for 1995 publication.

[P1] Jean Paul Pemba, *A set-theoretic characterization of I-Saturated James Type spaces,* Acta Mathematica H., - to be published in vol. 67 (2), pp. 215-228, Budapest, 1995.

[Y] Abdul-Aziz Yakubu, *The effects of planting and harvesting and endangered species in discrete competitive systems,* Mathematical Biosciences, vol 126, pp 1-20, 1995.

II.B. SUBMITTED

[GK] David Gurarie and Mark Kon, *Sobolev Smoothing by Functions of Elliptic Operators.*

[K] Rodney Kerby, *The Weiner-Wintner Theerem in Higher Dimension,* Journal of Harmonic Analysis and Its Applications.

[P2] Jean Paul Pemba, *Quasiorthonormailty and Wavelets,* J. Functional Analysis & Applications.

[P3] Jean-Paul Pemba, *Structure Theory of James Type Spaces,* Illinois J. of Mathematics.

II.C. IN PREPARATION

[KR3] Mark Kon and Louise Raphael, *Error Bounds and Convergence Rates of Wavelets Expansions*

[P4] Jean Paul Pemba, *Gram-Schmidt Operators and Wavelets,*

III. PROFESSIONAL PERSONNEL AND CONSULTANTS

A. FULL TIME.

1. Harvey Diamond, Professor of Mathematics, West Virginia University
2. Rodney Kerby, Assistant Professor of Mathematics, Howard University
3. Mark Kon, Associate Professor of Mathematics, Boston University
4. Jean-Paul Pemba, Post-Doctorate, Howard University
5. Louise Raphael, Professor of Mathematics and Principal Investigator, Howard University
6. Daniel Williams, Lecturer and Co-Principal Investigator, Howard University.

PART TIME RESEARCHERS

7. Aziz Yakubu, Associate Professor of Mathematics

III.B.1. PRINCIPAL CONSULTANTS

a. John Benedetto, University of Maryland

John Benedetto is a leading authority on harmonic analysis and wavelets. Benedetto is directing the Howard University's noise classification project.

b. Charles, Chui, Texas A & M University

Charles Chui is a leading authority on spline approximation and wavelet analysis. Chui will assist us in our research employing spline-wavelets, including image compression algorithms. Both consultants have suggested further related problems involving wavelets, and act as a resource providing up-to-date knowledge in this fast moving field.

III.B.2. OTHER CONSULTANTS

c. Archil Gulisashvilli, Boston University and Cornell University

d. Wim Sweldens, Catholic University, Leuven, Belgium and University of South Carolina

e. Kosmo Tatalias, AHPCRC

III.C. GRADUATE STUDENTS

PH.D.

1. Amatleleah Al-hoori, 1994
2. Xiaoming Fu, M.S. and candiate for Ph.D.

M.S.

3. Ingrid Brown, 1994
4. Robin Blalock
5. Nicole Williams
6. Shauna Anderson
7. Jackie Symms
8. Samuel Baffoe

IV. PAPERS PRESENTED AT CONFERENCES AND SEMINARS

* presenter

WAVELETS

1. L. Raphael, Eighth International Approximation theory Conference, College Station, TX, Jan. 1995.
2. M. Kon and L. Raphael*, Special Semester in Approximation Theory, The Technion, Haifa, Israel, June 1994.
3. M. Kon and L. Raphael*, Mathematics Institute, University of Aachen, Aachen. Germany, June 1994.
4. M. Kon and L. Raphael*, Steklov Mathematics Institute, Moscow, Russia, June 1994.
5. M. Kon and L. Raphael*, Neaman Workshop in Signal Processing, The Technion, Haifa, Israel, May 1994.
6. L. Raphael*, Weizmann Institute of Science, Rehovot, Israel, May 1994.
7. M. Kon* and L. Raphael, AMS Special Session on Wavelets, Cincinnati, OH, January 1994.
8. M. Kon and L. Raphael*, International Conference on Signal Processing and Mathematical Analysis, Cairo, Egypt, January 1994.

MATHEMATICAL PHYSICS

28. M. Kon, Mathematics Department, Harvard University, May 1993.

V. SELECTIVE WAVELETS SEMINARS/COLLOQUIUMS at HOWARD UNIVERSITY

1. Howard University wavelets group consisting of a core of three faculty members and three graduate students held weekly seminars. Sept 1991- June 1992.

2. John Benedetto, University of Maryland: "Sampling Theorems and Wavelets", October 1991.

3. Gilbert Strang, MIT: "Wavelets and Dilation Equations", November 1991.

4. Wim Sweldens, Katholieke Universiteit, Belgium; 3 seminars on "Applications of Wavelets", .May 1992.

5. Wally Maydych, University of Connecticut. "Wavelets", October 1992.

6. John Benedetto: several meetings at Howard and UMD on "Classification of Noise", April - September 1993.

7. Ingrid Daubechies, "Wavelets: What is all the fuss about?", April 1993.